6 INEQUALITIES

6.1. Definitions and elementary properties.

A number a is said to be *positive*, if a be greater than zero, and is written as $a > 0$.

A number a is said to be *negative*, if a be less than zero, and is written as $a < 0$.

The square of a real number is positive.

For any two real numbers a and b, either $a = b$, or $a > b$, or $a < b$. The first one is *equality* and the last two are *inequalities*.

Hence $a \neq b$ implies that either $a > b$ or $a < b$.

Thus there are two types of inequalities, 'less than ($<$)' type and 'greater than ($>$)' type.

A quantity a is said to be greater than another quantity b, if $(a - b)$ be positive, and is written as $a > b$. Thus (-2) is greater than (-3), since $(-2) - (-3) = -2 + 3 = 1$ which is positive. $a > b$ implies that $b < a$.

A quantity a is said to be less than another quantity b, if $(a - b)$ be negative, and is written as $a < b$. $a < b$ implies that $b > a$.

The symbol '$\leq$' means 'either less than or equal to' and the symbol '$\geq$' means 'either greater than or equal to'.

$a \ngtr b$ implies that $a \leq b$ and $a \nless b$ implies that $a \geq b$.

For the real and positive quantities a, b, c, the following properties follow directly :

(*a*) If $a > b$, then (*i*) $a + c > b + c$,

(*ii*) $a - c > b - c$,

(*iii*) $ac > bc$,

(*iv*) $\frac{a}{c} > \frac{b}{c}$,

(*v*) $-ac < -bc$,

(*vi*) $a^{n} > b^{n}$,

(*vii*) $a^{-n} < b^{-n}$

and, in particular, (*viii*) $-a < -b$ and $\frac{1}{a} < \frac{1}{b}$.

(b) If $a > b > c$, that is, if $a > b$ and $b > c$, then $a > c$.

(c) If $a > b, c > d, e > f, \ldots\ldots$, then

$$a + c + e + \ldots\ldots > b + d + f + \ldots\ldots$$

and

$$a c e \ldots\ldots > b d f \ldots\ldots$$

(d) If both sides of an inequality be symmetric functions of $a, b, c, \ldots\ldots$, then it may be assumed that

$$a \geq b \geq c \geq \ldots..$$

If the variable quantity x satisfies the condition $x \leq k$, where k is a constant, then the maximum value of x is k. If $x \geq k$, then the minimum value of x is k.

In this chapter, unless otherwise stated, the numbers used are assumed to be real and n is assumed to be a positive integer greater than 1.

6.2. Illustrative Examples.

Ex. 1. *If a, b, c be any three real numbers, then show that*

(i) $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(ii) $\dfrac{b^2 + c^2}{b + c} + \dfrac{c^2 + a^2}{c + a} + \dfrac{a^2 + b^2}{a + b} \geq a + b + c$. [*B. H. 1987, 1997*]

In the second case, the three numbers are positive.

(i) Here $a^2 + b^2 + c^2 - (ab + bc + ca)$

$$= \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq 0.$$

Therefore $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(ii) Here $\dfrac{b^2 + c^2}{b + c} + \dfrac{c^2 + a^2}{c + a} + \dfrac{a^2 + b^2}{a + b} - (a + b + c)$

$$= \left(\frac{b^2 + c^2}{b + c} - \frac{b + c}{2}\right) + \left(\frac{c^2 + a^2}{c + a} - \frac{c + a}{2}\right) + \left(\frac{a^2 + b^2}{a + b} - \frac{a + b}{2}\right)$$

$$= \frac{(b - c)^2}{2(b + c)} + \frac{(c - a)^2}{2(c + a)} + \frac{(a - b)^2}{2(a + b)} \geq 0$$ since a, b, c are positive.

Therefore $\dfrac{b^2 + c^2}{b + c} + \dfrac{c^2 + a^2}{c + a} + \dfrac{a^2 + b^2}{a + b} \geq a + b + c$.

Ex. 2. (a) *If any two of a, b, c be together greater than the third and $x + y + z = 0$, then show that*

$$a^2 yz + b^2 zx + c^2 xy \leq 0$$ [*C. H. 1977; V. H. 1988*]

$$(x \neq 0, y \neq 0, z \neq 0).$$

(b) *If a, b, x, y be all positive, then show that*

$$\frac{(a + b)\,xy}{ay + bx} \leq \frac{ax + by}{a + b}.$$ [*C. H. 1980, 1982*]

(a) As $x+y+z=0$, it may be assumed that x and y are positive and z is negative. Thus $z=-(x+y)$.

Now $a^2yz+b^2zx+c^2xy$

$$=(a^2y+b^2x)z+c^2xy$$

$$=-(a^2y+b^2x)(x+y)+c^2xy$$

$$=-xy\{(a+b)^2-c^2\}-(ay-bx)^2\le 0,$$

since $\{(a+b)^2-c^2\}$ is positive, as $a+b>c$.

(b) Here $\dfrac{(a+b)xy}{ay+bx}-\dfrac{ax+by}{a+b}$

$$=\frac{(a+b)^2xy-(ax+by)(ay+bx)}{(a+b)(ay+bx)}$$

$$=-\frac{ab(x-y)^2}{(a+b)(ay+bx)}\le 0,$$ since a, b, x, y are all positive.

Therefore $\dfrac{(a+b)xy}{ay+bx}\le\dfrac{ax+by}{a+b}$.

Ex. 3. *If each of a, b, c, d be greater than 1, then show that*

$$8(abcd+1)>(a+1)(b+1)(c+1)(d+1).$$ [T. H. 2008]

We have $2(ab+1)-(a+1)(b+1)$

$$=ab-a-b+1$$

$$=(a-1)(b-1)>0,$$ since $a>1$ and $b>1$.

Therefore $(a+1)(b+1)<2(ab+1)$.

Similarly, $(c+1)(d+1)<2(cd+1)$.

Therefore $(a+1)(b+1)(c+1)(d+1)<4(ab+1)(cd+1)$

$$<4.2(ab.cd+1)$$

$$<8(abcd+1).$$

Hence $8(abcd+1)>(a+1)(b+1)(c+1)(d+1)$.

Ex. 4. *Prove that* $\dfrac{1}{2\sqrt{n+1}}<\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdots\cdots\dfrac{2n-1}{2n}<\dfrac{1}{\sqrt{2n+1}}$.

[B. H. 1969; C. H. 1979]

Let $u_n=\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdots\cdots\dfrac{2n-1}{2n}$.

Now $\dfrac{1}{2}<\dfrac{2}{3}$, $\dfrac{3}{4}<\dfrac{4}{5}$, $\dfrac{5}{6}<\dfrac{6}{7}$, ..., $\dfrac{2n-1}{2n}<\dfrac{2n}{2n+1}$.

Combining them, $\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdots\cdots\dfrac{2n-1}{2n}<\dfrac{2}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{7}\cdots\cdots\dfrac{2n}{2n+1}$

or, $u_n<\dfrac{2}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{7}\cdots\cdots\dfrac{2n}{2n+1}$.

Therefore $u_n^{\ 2} = u_n \,.\, u_n < \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n}\right)\left(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n+1}\right)$

$< \frac{1}{2n+1}$

or, $u_n < \frac{1}{\sqrt{2n+1}}$ . ... (1)

Also $(2n+1)\,u_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n}\cdot(2n+1)$

$= \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\cdots\frac{2n+1}{2n}$.

Now $\frac{3}{2} > \frac{4}{3}$, $\frac{5}{4} > \frac{6}{5}$, $\frac{7}{6} > \frac{8}{7}$, ..., $\frac{2n+1}{2n} > \frac{2n+2}{2n+1}$.

Combining them, $\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\cdots\frac{2n+1}{2n} > \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\cdots\frac{2n+2}{2n+1}$

or, $(2n+1)\,u_n > \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\cdots\frac{2n+2}{2n+1}$.

Therefore $(2n+1)^2\,u_n^{\ 2} = (2n+1)\,u_n\,.\,(2n+1)\,u_n$

$> \left(\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\cdots\frac{2n+1}{2n}\right)\left(\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\cdots\frac{2n+2}{2n+1}\right)$

$> n+1$

or, $(2n+1)\,u_n > \sqrt{n+1}$

or, $u_n > \frac{\sqrt{n+1}}{2n+1} > \frac{\sqrt{n+1}}{2n+2}$

or, $u_n > \frac{1}{2\sqrt{n+1}}$ . ... (2)

The result follows from (1) and (2) .

Ex. 5. *Prove that* $1\,!.\,3\,!.\,5\,!\,......\,(2n-1)\,! > (n\,!)^n$.

[C. H. 1980, 1981]

If $n > r$, then we have $2n - r > n$; (r being a positive integer).

Then $n\,! = 1\,.\,2\,.\,3\,.\,......\,r\,(r+1)\,(r+2)\,......\,(n-1)\,n$

$= r\,!\,(r+1)\,(r+2)\,......\,(n-1)\,n$

and $(2n-r)\,! = 1.\,2.\,3.\,......\,n\,(n+1)\,(n+2)\,......\,(2n-r)$

$= n\,!\,(n+1)\,(n+2)\,.......\,(2n-r)$.

Therefore $\frac{r\,!}{n\,!}\cdot\frac{(2n-r)\,!}{n\,!} = \frac{(n+1)\,(n+2)\,......\,(2n-r)}{(r+1)\,(r+2)\,.....\,n} > 1$.

Hence $r\,!\,(2n-r)\,! > (n\,!)^2$.

If $r > n$, then we have $2n - r < n$.

Then $r! = 1.2.3. \ldots\ldots n(n+1)(n+2) \ldots\ldots r$

$= n!(n+1)(n+2) \ldots\ldots r$

and $n! = 1.2.3. \ldots\ldots (2n-r)(2n-r+1) \ldots\ldots n$

$= (2n-r)!(2n-r+1) \ldots\ldots n.$

Therefore $\frac{r!}{n!} \cdot \frac{(2n-r)!}{n!} = \frac{(n+1)(n+2) \ldots\ldots r}{(2n-r+1)(2n-r+2) \ldots\ldots n} > 1.$

Hence, in this case also, $r!(2n-r)! > (n!)^2$.

Putting 1, 3, 5,......, $(2n-1)$ successively for r, we get

$$1!(2n-1)! > (n!)^2,$$
$$3!(2n-3)! > (n!)^2,$$
$$5!(2n-5)! > (n!)^2,$$

....

....

....

$$(2n-1)!1! > (n!)^2.$$

Multiplying together, we get

$$\{1!3!5! \ldots\ldots (2n-1)!\}^2 > (n!)^{2n}.$$

Taking the positive square root of both sides, we obtain the result.

Ex. 6. *Find the minimum value of* $\frac{(a+x)(b+x)}{c+x}$.

Let $c + x = y$.

Then $\frac{(a+x)(b+x)}{c+x} = \frac{(a-c+y)(b-c+y)}{y}$

$$= y + (a-c) + (b-c) + \frac{(a-c)(b-c)}{y}$$

$$= \left\{ \sqrt{y} - \frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}} \right\}^2 + \{\sqrt{a-c} + \sqrt{b-c}\}^2$$

which is minimum when $\sqrt{y} - \sqrt{\frac{(a-c)(b-c)}{y}} = 0$

that is, when $y = \sqrt{(a-c)(b-c)}$

that is, when $x = \sqrt{(a-c)(b-c)} - c$

and the minimum value of the given expression is

$$(\sqrt{a-c} + \sqrt{b-c})^2.$$

Examples VI (A)

1. If a, b, c be any real numbers, then show that

(i) $(b+c-a)^2+(c+a-b)^2+(a+b-c)^2 \geq bc+ca+ab$.

(ii) $b^2c^2+c^2a^2+a^2b^2 \nless abc(a+b+c)$.

2. If a, b, c, be any three positive real numbers, then prove that

(i) $\dfrac{b+c}{b^2+c^2}+\dfrac{c+a}{c^2+a^2}+\dfrac{a+b}{a^2+b^2} \leq \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$. [K. H. 1979]

(ii) $\dfrac{bc}{b+c}+\dfrac{ca}{c+a}+\dfrac{ab}{a+b} \leq \frac{1}{2}(a+b+c)$.

(iii) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \nless \dfrac{1}{\sqrt{bc}}+\dfrac{1}{\sqrt{ca}}+\dfrac{1}{\sqrt{ab}}$.

3. Show that

(i) $a^3+b^3>a^2b+ab^2$.

(ii) $x^6+y^6 \nless x^5y+xy^5$.

(iii) $2(a^9+b^9)>(a^5+b^5)(a^4+b^4)$.

(iv) $(a^2+b^2+1)(c^2+d^2+1) \nless (ac+bd+1)^2$.

(v) $a^2+b^2+c^2+d^2>ab+bc+cd+da$. [B. H. 2002]

(vi) $a^2+b^2+c^2+d^2 \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$. [B. H. 2005]

4. If a, b, x be positive, then

(i) show that $\dfrac{a+x}{b+x}>\dfrac{a}{b}$, if $a<b$;

(ii) find under what condition $\dfrac{a+x}{b+x}<\dfrac{a}{b}$.

5. Under what circumstances $x^3+25x<8x^2+26$?

6. Show that $\dfrac{a-x}{a+x}<\dfrac{a^2-x^2}{a^2+x^2}$, if $a>x>0$.

7. If $a>0$ and $x>y$, then show that $a^x+a^{-x}>a^y+a^{-y}$.

8. If $x>y>z$, then show that

$$x^4y+y^4z+z^4x>xy^4+yz^4+zx^4$$

9. Prove that $(a^2+b^2+c^2)(x^2+y^2+z^2) \geq (ax+by+cz)^2$ and hence deduce that $ax+by+cz \leq 1$, if $a^2+b^2+c^2=1$ and $x^2+y^2+z^2=1$.

10. If the four positive numbers a, b, c, d be in H.P., then show that $a+d>b+c$.

6.3. Theorem on A.M., G.M. and H.M.

The arithmetic mean (A.M.) of the n positive numbers

$$a_1, a_2, a_3, \ldots\ldots, a_n$$

is $$A = \frac{a_1 + a_2 + a_3 + \cdots\cdots + a_n}{n}.$$

Their geometric mean (G. M.) is

$$G = \sqrt[n]{a_1 a_2 a_3 \ldots\ldots a_n}.$$

and their harmonic mean (H.M.) is

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots\cdots + \frac{1}{a_n}}.$$

The relation among A, G and H is $A \geq G \geq H$, the sign of equality occurs when all the numbers are equal.

For positive values of a_1 and a_2, it follows that

$$(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$$

$$a_1 + a_2 - 2\sqrt{a_1 a_2} \geq 0$$

or,

$$\tfrac{1}{2}(a_1 + a_2) \geq \sqrt{a_1 a_2}$$

or,

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2}\right)^2.$$

or,

The sign of equality occurs when $a_1 = a_2$.

Similarly, for positive values of a_3 and a_4, $a_3 a_4 \leq \left(\frac{a_3 + a_4}{2}\right)^2$.

The sign of equality occurs when $a_3 = a_4$.

$$\text{Therefore } a_1 a_2 a_3 a_4 \leq \left(\frac{a_1 + a_2}{2}\right)^2 \left(\frac{a_3 + a_4}{2}\right)^2$$

$$\leq \left(\frac{a_1 + a_2}{2} \cdot \frac{a_3 + a_4}{2}\right)^2$$

$$\leq \left\{\left(\frac{\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2}}{2}\right)^2\right\}^2$$

$$\leq \left(\frac{a_1 + a_2 + a_3 + a_4}{4}\right)^4,$$

the sign of equality occurs when $a_1 = a_2 = a_3 = a_4$.

Proceeding in this way, it may be obtained that

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 \leq \left(\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8}{8}\right)^8,$$

the sign of equality occurs when $a_1 = a_2 = a_3 = \cdots\cdots = a_8$.

Thus, if n be a power of 2, that is, if $n = 2^m$, m being a positive integer, then we get

$$a_1 a_2 a_3 \ldots\ldots a_n \leq \left(\frac{a_1 + a_2 + a_3 + \cdots\cdots + a_n}{n}\right)^n, \qquad \ldots \quad (1)$$

the sign of equality occurs when $a_1 = a_2 = a_3 = \cdots\cdots = a_n$.

If n be not a power of 2, then let $(n + p)$ be a power of 2, where p is a positive integer.

Now consider the $(n+p)$ positive numbers $a_1, a_2, a_3, \ldots\ldots, a_n$ and p numbers each equal to a, where

$$a = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}. \qquad \ldots (2)$$

From (1), it follows that

$$a_1 a_2 a_3 \ldots\ldots a_n a^p \le \left(\frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n + pa}{n+p}\right)^{n+p}$$

$$\le \left(\frac{na + pa}{n+p}\right)^{n+p}, \text{ from (2)}$$

$$\le a^{n+p}$$

or, $a_1 a_2 a_3 \ldots\ldots a_n \le a^n$

$$\le \left(\frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}\right)^n,$$

the sign of equality occurs when $a_1 = a_2 = a_3 = \ldots\ldots = a_n$.

Therefore $\frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n} \ge \sqrt[n]{a_1 a_2 a_3 \ldots\ldots a_n}$

or, $A \ge G$, the sign of equality occurs when all numbers are equal.

To prove $G \le H$, consider the positive numbers

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots\ldots, \frac{1}{a_n}.$$

Their A.M. ≥ their G.M.

Therefore $$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_n}}{n} \ge \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdot \frac{1}{a_3} \cdot \ldots\ldots \frac{1}{a_n}},$$

the sign of equality occurs when $\frac{1}{a_1} = \frac{1}{a_2} = \frac{1}{a_3} = \ldots\ldots = \frac{1}{a_n}$,

that is, when $a_1 = a_2 = a_3 = \ldots\ldots = a_n$.

Therefore $$\sqrt[n]{a_1 a_2 a_3 \ldots\ldots a_n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_n}}$$

or, $G \ge H$, the sign of equality occurs when all numbers are equal.

Hence $A \ge G \ge H$, equality occurs when all the numbers are equal.

Note. The inequality $A > G$ for unequal numbers may also be proved in the following way :

In the set of n positive numbers $\{a_1, a_2, a_3, \ldots, a_n\}$, there should be at least one number less than G and one number greater than G. Let $a_r > G$ and $a_s < G$. We replace the first set by a new set $\{a_1', a_2', a_3', \ldots, a_n'\}$ such that $a_r' = G$ and $a_s' = \frac{a_r a_s}{G}$ and $a_n' = a_n$ for all n except $n = r, s$.

If A' and G' be the respective A.M. and G.M. of the new set, then

$$n(A - A') = a_r + a_s - a_r' - a_s' = (a_r - G) + a_s - \frac{a_r a_s}{G}$$

$$= \frac{(a_r - G)(G - a_s)}{G} > 0.$$

Therefore $A > A'$.

But $G' = \sqrt[n]{a_1' a_2' \cdots a_r' \cdots a_s' \cdots a_n'}$

$$= \sqrt[n]{a_1 a_2 \cdots G \cdots \frac{a_r a_s}{G} \cdots a_n} = G.$$

Thus,in the new set,A.M. is decreased and G.M. is unaltered, one number being equal to G. We repeat the process of replacing each set by a new set of positive numbers,such that their A.M. is decreased and G.M. is unaltered. Proceeding in this way, after $(n-1)$ steps, we shall arrive at a set where all the numbers are equal to G ; because, after $(n-1)$ steps there will be $(n-1)$ numbers equal to G and the n-th number will also be G, as their G.M. is G.

If $A^{(n)}$ and $G^{(n)}$ be the A.M. and G.M. for the last set, then

$$A > A' > A'' > \ldots\ldots > A^{(n)}.$$

But $G = G' = G'' = \cdots\cdots = G^{(n)}$.

Also $A^{(n)} = G^{(n)}$, as all numbers are equal.

Therefore $A > A^{(n)}$

$> G^{(n)}$

$> G.$

It should be noted that this proof is also applicable when some numbers are equal ; because, if r numbers be equal, then after $(r-1)$ steps we shall arrive at a set where all numbers are equal to G.

6.4. Extreme values of sum and product.

(a) *If the product of n positive numbers be constant, then their sum is minimum when the numbers are all equal.*

Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be n positive numbers,such that

$$a_1 \, a_2 \, a_3 \, \ldots\ldots \, a_n = \text{constant} = k \text{ (say)}.$$

Their A. M. ≥ their G. M.

or, $$\frac{a_1+a_2+a_3+\cdots\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots\cdots a_n}$$

or, $$a_1+a_2+a_3+\cdots\cdots+a_n \geq n\sqrt[n]{k},$$

the sign of equality occurs when $a_1=a_2=a_3=\cdots\cdots=a_n$.

Therefore the minimum value of $(a_1+a_2+a_3+\cdots\cdots+a_n)$ is $n\sqrt[n]{k}$ and it occurs when the numbers are all equal.

(*b*) *If the sum of n positive numbers be constant, then their product is maximum when the numbers are all equal.*

Let $a_1, a_2, a_3, \cdots\cdots, a_n$ be n positive numbers, such that
$$a_1+a_2+a_3+\cdots\cdots+a_n = \text{constant} = k \text{ (say).}$$

From the relation G.M. ≤ A.M., we get

$$\sqrt[n]{a_1 a_2 a_3 \cdots\cdots a_n} \leq \frac{a_1+a_2+a_3+\cdots\cdots+a_n}{n}$$

or, $$a_1 a_2 a_3 \cdots\cdots a_n \leq \left(\frac{k}{n}\right)^n,$$

the sign of equality occurs when $a_1=a_2=a_3=\cdots\cdots=a_n$.

Therefore the maximum value of $a_1 a_2 a_3 \cdots\cdots a_n$ is $\left(\frac{k}{n}\right)^n$ and it occurs when the numbers are all equal.

6.5. Theorem of weighted means.

If $a_1, a_2, a_3, \cdots\cdots, a_n$ *and* $x_1, x_2, x_3, \cdots\cdots, x_n$ *be two sets of n positive numbers, those of the second set being rational, then*

$$\left(\frac{a_1x_1+a_2x_2+a_3x_3+\cdots\cdots+a_nx_n}{x_1+x_2+x_3+\cdots\cdots+x_n}\right)^{x_1+x_2+x_3+\cdots\cdots+x_n}$$
$$\geq a_1^{x_1}.a_2^{x_2}.a_3^{x_3}\cdots\cdots a_n^{x_n},$$

the sign of equality occurs when $a_1=a_2=a_3=\cdots\cdots=a_n$.

As $x_1, x_2, x_3, \cdots\cdots, x_n$ are positive rational numbers, there exist positive integers $x_1', x_2', x_3', \cdots\cdots, x_n', g$, such that

$$x_1=\frac{x_1'}{g},\ x_2=\frac{x_2'}{g},\ x_3=\frac{x_3'}{g},\ \cdots\cdots,\ x_n=\frac{x_n'}{g}.$$

Now consider

x_1' numbers each equal to a_1,

x_2' .... a_2,

x_3' .... a_3,

....

....;

x_n' .... a_n.

Their A. M. ≥ their G.M.

or,
$$\frac{a_1 x_1' + a_2 x_2' + a_3 x_3' + \cdots\cdots + a_n x_n'}{x_1' + x_2' + x_3' + \cdots\cdots + x_n'}$$

$$\geq \left(a_1^{x_1'} . a_2^{x_2'} . a_3^{x_3'} \cdots\cdots a_n^{x_n'}\right)^{\frac{1}{x_1' + x_2' + x_3' + \cdots\cdots + x_n'}}$$

The sign of equality occurs when $a_1 = a_2 = a_3 = \cdots\cdots = a_n$.

Putting $x_1' = x_1 g$, $x_2' = x_2 g$, $x_3' = x_3 g$, ..., $x_n' = x_n g$ in both sides of the inequality, we obtain

$$\frac{a_1 x_1 g + a_2 x_2 g + a_3 x_3 g + \cdots\cdots + a_n x_n g}{x_1 g + x_2 g + x_3 g + \cdots\cdots + x_n g}$$

$$\geq \left(a_1^{x_1 g} . a_2^{x_2 g} . a_3^{x_3 g} \cdots\cdots a_n^{x_n g}\right)^{\frac{1}{x_1 g + x_2 g + x_3 g + \cdots\cdots + x_n g}}$$

or,
$$\frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n}{x_1 + x_2 + x_3 + \cdots\cdots + x_n}$$

$$\geq \left(a_1^{x_1} . a_2^{x_2} . a_3^{x_3} \cdots\cdots a_n^{x_n}\right)^{\frac{1}{x_1 + x_2 + x_3 + \cdots\cdots + x_n}}$$

or,
$$\left(\frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n}{x_1 + x_2 + x_3 + \cdots\cdots + x_n}\right)^{x_1 + x_2 + x_3 + \cdots\cdots + x_n}$$

$$\geq a_1^{x_1} . a_2^{x_2} . a_3^{x_3} \cdots\cdots a_n^{x_n}.$$

The sign of equality occurs when $a_1 = a_2 = a_3 = \cdots\cdots = a_n$.

6.6. Illustrative Examples.

Ex.1. *If a, b, c be positive, then show that*

$$(bc + ca + ab)(a^4 + b^4 + c^4) > 9a^2b^2c^2.$$

Let us consider the three positive quantities bc, ca, ab.

Their A.M. > their G.M.

Therefore $\frac{1}{3}(bc+ca+ab) > \sqrt[3]{bc\,.\,ca\,.\,ab}$

or, $\quad bc+ca+ab > 3(abc)^{\frac{2}{3}}$. $\quad\ldots$ (1)

Again let us consider the three positive quantities a^4, b^4, c^4.

Their A.M. > their G.M.

Therefore $\frac{1}{3}(a^4+b^4+c^4) > \sqrt[3]{a^4 b^4 c^4}$

or, $\quad a^4+b^4+c^4 > 3(abc)^{\frac{4}{3}}$. $\quad\ldots$ (2)

Multiplying (1) and (2), we get the result.

Ex. 2. (*a*) *If a, b, c be positive, then prove that*

$$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} > \frac{9}{a+b+c},$$

unless $a=b=c$. [C. H. 1970]

(*b*) *If* a_1, a_2, a_3, a_4 *be distinct positive numbers and* $s = a_1+a_2+a_3+a_4$, *then show that*

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\frac{s}{s-a_3}+\frac{s}{s-a_4} > 5\tfrac{1}{3}.$$ [C. H. 1980]

(*a*) Let us consider the three positive quantities

$$\frac{2}{b+c}, \frac{2}{c+a}, \frac{2}{a+b}.$$

Their A.M $\geq$ their H.M.

Therefore $$\frac{\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}}{3} \geq \frac{3}{\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}}$$

or, $$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} \geq \frac{9}{a+b+c}.$$

The sign of equality occurs when $\frac{2}{b+c}=\frac{2}{c+a}=\frac{2}{a+b}$,

that is, when $b+c=c+a=a+b$, that is, when $a=b=c$.

Therefore $$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} > \frac{9}{a+b+c}$$

unless $a=b=c$, in which case $\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} = \frac{9}{a+b+c}$.

(*b*) Since a_1, a_2, a_3, a_4 are distinct positive numbers and $s = a_1 + a_2 + a_3 + a_4$, therefore the four numbers

$$\frac{s}{s-a_1}, \frac{s}{s-a_2}, \frac{s}{s-a_3}, \frac{s}{s-a_4}$$ are also distinct and positive.

Using A.M. > H.M. for the second set of positive numbers, we get

$$\frac{\frac{s}{s-a_1}+\frac{s}{s-a_2}+\frac{s}{s-a_3}+\frac{s}{s-a_4}}{4} > \frac{4}{\frac{s-a_1}{s}+\frac{s-a_2}{s}+\frac{s-a_3}{s}+\frac{s-a_4}{s}}$$

or, $$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\frac{s}{s-a_3}+\frac{s}{s-a_4} > \frac{16s}{4s-(a_1+a_2+a_3+a_4)}$$

$$> \frac{16s}{4s-s}, \text{ that is, } > \frac{16}{3}$$

or, $$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\frac{s}{s-a_3}+\frac{s}{s-a_4} > 5\tfrac{1}{3}.$$

Ex. 3. *Show that* $(n+1)^n > 2^n . n!$. [*K. H. 1979; B. H. 1982*]

Let us consider the n distinct positive numbers $1, 2, 3, \ldots\ldots, n$.

Their A.M . > their G.M .

Therefore $$\frac{1+2+3+\cdots\cdots+n}{n} > \sqrt[n]{1.2.3\ldots\ldots n}$$

or, $$\frac{n(n+1)}{2n} > (n!)^{\frac{1}{n}}$$

or, $$n+1 > 2(n!)^{\frac{1}{n}}$$

or, $$(n+1)^n > 2^n . n!.$$

Ex. 4. *If* a_1, a_2, a_3, a_4, a_5 *be all positive, then prove that*

$$\left(\frac{a_1+a_2+a_3+a_4+a_5}{5}\right)^5 \geq \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3.$$ [C. H. 1986]

Let us consider 2 numbers each equal to $\frac{a_1+a_2}{2}$

and 3 numbers each equal to $\frac{a_3+a_4+a_5}{3}$.

Using A.M . ≥ G.M. for these 5 positive numbers, we get

$$\frac{2\times\frac{a_1+a_2}{2}+3\times\frac{a_3+a_4+a_5}{3}}{5} \geq \sqrt[5]{\left(\frac{a_1+a_2}{2}\right)^2\left(\frac{a_3+a_4+a_5}{3}\right)^3}$$

or, $$\left(\frac{a_1+a_2+a_3+a_4+a_5}{5}\right)^5 \geq \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3$$

Ex. 5. *If $a_1, a_2, a_3, \ldots\ldots, a_n$ be n positive numbers and $a_n a_{n-1} = 1$ then show that*

$$\left(\frac{a_1+a_2+a_3+\cdots\cdots+a_n}{n}\right)^n \geq \left(\frac{a_1+a_2+a_3+\cdots\cdots+a_{n-2}}{n-2}\right)^{n-2}.$$ [C.H.1969]

Let us consider the set

$$a, a_{n-1}, a_n, \text{ where } a = \frac{a_1+a_2+a_3+\cdots\cdots+a_{n-2}}{n-2}$$

with weights $(n-2), 1, 1$ respectively.

Applying A. M. $\geq$ G. M., we get

$$\frac{(n-2)a+a_{n-1}+a_n}{(n-2)+1+1} \geq \sqrt[n]{a^{n-2}.a_{n-1}.a_n}$$

or, $$\frac{a_1+a_2+a_3+\cdots\cdots+a_n}{n} \geq \sqrt[n]{a^{n-2}}, \text{ since } a_n a_{n-1} = 1$$

or, $$\left(\frac{a_1+a_2+a_3+\cdots\cdots+a_n}{n}\right)^n \geq \left(\frac{a_1+a_2+a_3+\cdots\cdots+a_{n-2}}{n-2}\right)^{n-2}$$

Ex. 6. *If $a_1, a_2, a_3, \ldots\ldots, a_n$ be n positive rational numbers whose sum is s, then show that*

$$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\left(\frac{s}{a_3}-1\right)^{a_3}\cdots\cdots\left(\frac{s}{a_n}-1\right)^{a_n} \leq (n-1)^s.$$

[*C. H. 1969*]

Applying the theorem of weighted means to the set of n positive numbers $\left(\frac{s}{a_1}-1\right), \left(\frac{s}{a_2}-1\right), \left(\frac{s}{a_3}-1\right), \ldots\ldots, \left(\frac{s}{a_n}-1\right)$ with associated weights $a_1, a_2, a_3, \ldots\ldots, a_n$ respectively, we get

$$\left\{\frac{a_1\left(\frac{s}{a_1}-1\right)+a_2\left(\frac{s}{a_2}-1\right)+a_3\left(\frac{s}{a_3}-1\right)+\cdots\cdots+a_n\left(\frac{s}{a_n}-1\right)}{a_1+a_2+a_3+\cdots\cdots+a_n}\right\}^{a_1+a_2+a_3+\cdots\cdots+a_n}$$

$$\geq \left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\left(\frac{s}{a_3}-1\right)^{a_3}\cdots\cdots\left(\frac{s}{a_n}-1\right)^{a_n}$$

or, $$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\left(\frac{s}{a_3}-1\right)^{a_3}\cdots\cdots\left(\frac{s}{a_n}-1\right)^{a_n}$$

$$\leq \left(\frac{ns-s}{s}\right)^s, \text{ that is, } \leq (n-1)^s.$$

Ex. 7. *Find the minimum value of $(2x+3y)$ for positive values of x and y subject to the condition $12x^3y^4=1$.*

Since $x^3y^4=\frac{1}{12}$, if p, q be constants, we have

$$(px)(px)(px)(qy)(qy)(qy)(qy)=\tfrac{1}{12}p^3q^4=\text{constant}.$$

Therefore $px+px+px+qy+qy+qy+qy=3px+4qy$ is minimum when

$$px=qy=\left(\tfrac{1}{12}p^3q^4\right)^{\frac{1}{7}}.$$

Hence the minimum value of $(3px+4qy)$ is $7\left(\frac{1}{12}p^3q^4\right)^{\frac{1}{7}}$.

Putting $3p=2$ and $4q=3$, that is, $p=\frac{2}{3}$ and $q=\frac{3}{4}$, it follows that the minimum value of $(2x+3y)$ is

$$7\left(\frac{1}{12}\cdot\frac{2^3}{3^3}\cdot\frac{3^4}{4^4}\right)^{\frac{1}{7}}=\frac{7}{2}.$$

Ex. 8. *Find the maximum value of $xyz(d-ax-by-cz)$, where all the factors and a, b, c, d are positive.*

The expression $xyz(d-ax-by-cz)$ is maximum, when $(ax)(by)(cz)(d-ax-by-cz)$ is maximum.

Now the sum of the factors of $(ax)(by)(cz)(d-ax-by-cz)$ is

$$ax+by+cz+d-ax-by-cz=d=\text{constant}.$$

Therefore the product $(ax)(by)(cz)(d-ax-by-cz)$ is maximum when

$$ax=by=cz=d-ax-by-cz$$

that is, when $x=\dfrac{d}{4a}, y=\dfrac{d}{4b}, z=\dfrac{d}{4c}$.

Therefore the maximum value of $xyz(d-ax-by-cz)$ is

$$\frac{d}{4a}\cdot\frac{d}{4b}\cdot\frac{d}{4c}\cdot\frac{d}{4}=\frac{d^4}{256abc}.$$

Examples VI (B)

1. (*a*) If a, b, c be positive, then show that

(*i*) $a^3+b^3+c^3\geq 3abc$. [*B. H. 1988*]

(*ii*) $a^2b+b^2c+c^2a\geq 3abc$.

(*iii*) $(a+b+c)(a^2+b^2+c^2)\nless 9abc$.

(*b*) If a, b, c be positive and not all equal, then show that

(*i*) $(a+b+c)(bc+ca+ab)>9abc$. [*B. H. 1990*]

(*ii*) $\dfrac{b+c}{a}+\dfrac{c+a}{b}+\dfrac{a+b}{c}>6$.

(*iii*) $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}>\frac{3}{2}$. [*N. B. H. 2006*]